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Transport Properties of Kicked and Quasiperiodic Hamiltonians

S. De Bièvre[1] and G. Forni[2]

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We study transport properties of Schrödinger operators depending on one or more parameters. Examples include the kicked rotor and operators with quasiperiodic potentials. We show that the mean growth exponent of the kinetic energy in the kicked rotor and of the mean square displacement in quasiperiodic potentials is generically equal to 2: this means that the motion remains ballistic, at least in a weak sense, even away from the resonances of the models. Stronger results are obtained for a class of tight-binding Hamiltonians with an electric field $E(t) = E_0 + E_1 \cos \omega t$. For

$$H = \sum a_{n-k}(|n-k\rangle\langle n| + |n\rangle\langle n-k|) + E(t)\, |n\rangle\langle n|$$

with $a_n \sim |n|^{-\nu}$ ($\nu > 3/2$) we show that the mean square displacement satisfies $\overline{\langle \psi_t, N^2 \psi_t \rangle} \geqslant C_\varepsilon t^{2/(\nu+1/2)-\varepsilon}$ for suitable choices of ω, E_0, and E_1. We relate this behavior to the spectral properties of the Floquet operator of the problem.

KEY WORDS: Anomalous transport; singular spectra; kicked Hamiltonians.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

The asymptotic behaviour of time dependent quantities in the study of Schrödinger equations with quasi-periodic and random potentials, or in the study of periodically kicked or pulsed Hamiltonian systems has attracted much attention in the last decade. For a review we refer to [Ho] where many references beyond the ones cited below can be found. We present in

[1] UFR de Mathématiques et URA GAT, Université des Sciences et Technologies de Lille, 59655 Villeneuve d'Ascq Cedex, France; e-mail: debievre@gat.univ-lille1.fr.

[2] Department of Mathematics, Princeton University, Princeton, New Jersey 08544; e-mail: gforni@math.princeton.edu.

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Section 2 two simple abstract results stating roughly that "close to resonance" the spectrum of the Hamiltonian or the Floquet operator is continuous and that the motion has the same asymptotic characteristics as "in resonance," at least in some weak sense. In Section 3, we first apply these results to the kicked rotor. We give a simplified proof of the Casati-Guarneri result stating that generically the spectrum of the Floquet operator of the kicked rotor is continuous. We show in addition that the mean growth exponent of the time-averaged kinetic energy, defined below, generically equals 2 (Theorem 3.1). This can be paraphrased (somewhat exaggeratedly) by saying that the time-averaged kinetic energy generically behaves ballistically in the kicked rotor. We then show how to prove similar results for quasi-periodic Schrödinger operators. In Section 4, we study in detail the asymptotic behaviour of the kicked linear rotor. We exhibit various phenomena relating the asymptotic behaviour of the time-averaged kinetic energy and the nature of the spectrum of the Floquet operator, strenghtening known results. In Section 5 we then show how these results apply to the problem of the motion of a charged particle in a one-band tight-binding model subjected to a time-dependent electric field (Theorem 5.1). We now first describe our results and the motivations in some more detail.

Consider a periodically driven classical or quantal Hamiltonian system with Hamiltonian

$$H(t) = H_0 + V(t), \qquad V(t+T) = V(t)$$

In classical mechanics the evolution of the system is determined by the Floquet transformation Φ_V, obtained by integrating the Hamiltonian equations of motion over one period T. It is a canonical transformation of phase space M. Similarly, in quantum mechanics, integration of the Schrödinger equation over one period T yields the Floquet operator U_V, which is a unitary operator on the Hilbert space of states $\mathscr{H}$. Typically, the unperturbed classical Hamiltonian is chosen to be completely integrable with its motion restricted to invariant tori, so that in particular all its trajectories are bounded. The corresponding quantum Hamiltonian is assumed to have pure point spectrum so that the same is true for $U_0 = \exp -(i/h)\, TH_0$. What happens when V is turned on?

A natural first question to ask is whether the unperturbed energy H_0, which is a constant of the motion when $V=0$, remains bounded when $V \neq 0$. More precisely, the question in classical mechanics is whether

$$\sup_{m \in \mathbb{Z}} |H_0 \circ \Phi_V^m(x, p)| < \infty, \qquad (x, p) \in M$$

and in quantum mechanics whether

$$\sup_{m \in \mathbb{Z}} |\langle \psi, U_V^{-m} H_0 U_V^m \psi \rangle| < \infty, \qquad \psi \in \mathscr{D}(H_0)$$

where $\mathscr{D}(H_0)$ denotes the domain of H_0, assumed stable under U_V. Without giving a precise definition, let us say the system is dynamically stable in this case, and dynamically unstable otherwise.

In quantum mechanics one can ask a different, but related stability question [Be, Ho]: Is the spectrum of the Floquet operator U_V still pure point? We will say the system is spectrally stable if this is the case. It is a well known consequence of the RAGE theorem [CFKS] that dynamical stability implies spectral stability, the opposite implication not being true [DJLS].

We will be interested in dynamically unstable systems. Once the unperturbed energy does not remain bounded under the full, perturbed evolution, one can ask about its asymptotic behaviour: how do $|H_0 \circ \Phi_V^m(x, p)|$ and $|\langle \psi, U_V^{-m} H_0 U_V^m \psi \rangle|$ behave as m goes to infinity? One is in particular interested in finding out whether these quantities can have algebraic growth and, in the quantum mechanical case, to relate the growth exponent to the spectral properties of U_V.

Before reviewing the known results, let us quickly recall the various notions of growth exponent one might consider. Let h_m be a sequence of positive numbers. One says the sequence displays algebraic growth with exponent $\alpha(h) \geqslant 0$ if there exist constants c and C so that, for all $m \in \mathbb{N}^*$

$$c m^{\alpha(h)} \leqslant h_m \leqslant C m^{\alpha(h)} \tag{1.1}$$

It turns out this notion is too strong to be of use: the quantities of interest tend to fluctuate greatly as we will show in detail for the kicked linear rotor in Section 4. To obtain a fluctuation independent quantity, it has been suggested [G] one should consider a "mean growth exponent" $\alpha_0(h)$ defined as

$$\begin{aligned} \alpha_0(h) &= \inf \left\{ \alpha > 0 \,\middle|\, \sum_{m=1}^{\infty} \frac{1}{m^{1+\alpha}} h_m < \infty \right\} \\ &= \sup \left\{ \alpha > 0 \,\middle|\, \sum_{m=1}^{\infty} \frac{1}{m^{1+\alpha}} h_m = \infty \right\} \end{aligned} \tag{1.2}$$

where we use the (unusual) convention that $\inf \varnothing = \infty$ and $\sup \varnothing = 0$. One could alternatively decide to take the fluctuations into account, and introduce upper and lower growth exponents in the obvious way [G]:

$$\alpha_+(h) = \limsup_{m\to\infty} \frac{\log h_m}{\log m}, \qquad \alpha_-(h) = \liminf_{m\to\infty} \frac{\log h_m}{\log m} \tag{1.3}$$

Note that, even if $\alpha_+(h) = \alpha_-(h)$, (1.1) does not necessarily hold. On the other hand,

$$\alpha_-(h) \leqslant \alpha_0(h) \leqslant \alpha_+(h)$$

In addition, if (1.1) holds, $\alpha(h) = \alpha_+(h) = \alpha_-(h) = \alpha_0(h)$.

Let us write

$$\overline{\langle \psi, U_V^{-m} H_0 U_V^m \psi \rangle} = \frac{1}{m} \sum_{k=1}^{m} \langle \psi, U_V^{-k} H_0 U_V^k \psi \rangle$$

The problem is to get upper and lower bounds on this quantity. Upper bounds of the type

$$\overline{\langle \psi, U_V^{-m} H_0 U_V^m \psi \rangle} \leqslant C m^{\alpha} \tag{1.4}$$

were obtained in [N] under the assumption that $V(t)$ is smooth and the spectrum of H_0 satisfies a gap condition at infinity (see also [J]). The proofs are based on adiabatic techniques that do not allow for non-smooth time dependence and therefore do not apply to kicked systems. In addition, they do not provide lower bounds. One might hope to obtain lower bounds from an abstract approach initiated in [G] and perfected in [C, L, BCM]. Writing

$$U_V = \int \exp i\lambda \, dE_\lambda$$

for the spectral decomposition of U_V, the main result of this theory (Theorem 6.1 in [L], Theorem 3 in [BCM]) states the following. If the spectral measure $d\langle \psi, E_\lambda \psi \rangle \equiv d\mu_\psi(\lambda)$ has Hausdorff dimension β for some $0 \leqslant \beta \leqslant 1$ then there exists for all $\varepsilon > 0$ a constant C_ε, so that, for all $m \in \mathbb{N}^*$,

$$\overline{\langle \psi, U_V^{-m} H_0 U_V^m \psi \rangle} \geqslant C_\varepsilon m^{\gamma_{H_0}(\beta - \varepsilon)} \tag{1.5}$$

Here the constant γ_{H_0} depends on the spectral properties of H_0 in an explicit way [G]. The trouble with such lower bounds is that the information on the spectral measure is very hard to check: we do not know of any models where this has been done (other than numerically or for the trivial case where $\beta = 0$ or 1). As a result, we are not aware of any models where lower bounds of the type (1.5) (i.e., for all m) have been proven to hold for U_V with singular spectrum (other than numerically: see [G] for references). We will give such a model in Sections 4–5.

Actually, the only lower bound we know off is a result of Last ([L], Theorem 7.2) who shows that in the almost Mathieu equation there exist Liouville frequencies so that the quantity $\overline{\langle \psi_t, X^2 \psi_t \rangle}$ (with $\psi_0 = \delta_0$) has an upper growth exponent $\alpha_+ = 2$. We will strengthen this result in the following sense. We will show in Sections 3 that, not only in the Almost Mathieu equation (Theorem 3.2), but more generally in Schrödinger operators with quasi-periodic potentials as well as in the kicked rotor the relevant dynamical quantity has *generically* a *mean* growth exponent $\alpha_0 = 2$. The proof turns out to be very simple and is based on standard Baire theoretical arguments, sometimes irreverentially referred to as generic nonsense (Section 2). It seems therefore that the occurence of ballistic peaks close to resonance is a rather common phenomenon.

The growth exponents α_+ and α_0 only give lower bounds along a subsequence of times. To display a model in which non-trivial lower bounds *at all times* can be obtained, we present in Section 5 a tight-binding Hamiltonian with electric field $E(t) = E_0 + E_1 \cos \omega t$ and show that, for suitable choices of ω, $E_1 > E_0$ and provided the off-diagonal matrix elements satisfy $a_n \sim |n|^{-\nu}$ $(\nu > 3/2)$, one has for all ℓ and m

$$\overline{\langle \delta_\ell, U_T^{-m} N^2 U_T^m \delta_\ell \rangle} \geqslant C_\varepsilon m^{(2/(\nu + 1/2)) - \varepsilon}$$

where U_T is the Floquet operator of the theory (Theorem 5.1). This result is obtained by remarking that the model is equivalent to a kicked linear rotor and upon using results of [DBF]. A detailed analysis of the spectral properties of U_T is also given.

2. TWO GENERIC RESULTS

In many situations of interest either the Hamiltonian $H(\omega)$ or the Floquet operator $U(\omega)$ depend on one or several real variables in such a way that for a dense set of values of ω a "resonance" occurs. We will give examples in the next sections. For these values the spectrum is absolutely continuous and much is then known about the dynamical behaviour of the system. We show in this section two abstract results permitting to use this information to draw conclusions on the nature of the spectrum and on the asymptotic dynamical behaviour of the system for generic values of ω, "off resonance."

The term generic is used here in the topological sense. It means for ω belonging to a dense G_δ set. We recall that a G_δ set is a countable intersection of open sets and that, on $\mathbb{R}^n$, an intersection of two dense G_δ sets is still a dense G_δ set. In addition, dense G_δ sets are locally uncountable. It should nevertheless be stressed that dense G_δ sets can very well be of zero measure; in fact, in many applications where the set can be described explicitly, this is the case.

Let $\mathscr{H}$ be a Hilbert space and A a self-adjoint operator on $\mathscr{H}$ with domain $\mathscr{D}(A)$. Let $H(\omega)$ be a family of self-adjoint operators on $\mathscr{H}$ with common domain $\mathscr{D}$: we will assume throughout it is continuous in the strong resolvent sense. We will write $U_t(\omega) = \exp - itH(\omega)$ for the corresponding unitary one-parameter group, which is then strongly continuous, uniformly in t on compacta [RSI].

Proposition 2.1. Let $\mathscr{C} = \{\omega \mid \sigma_{pp}(H(\omega)) = \varnothing\}$. Then $\mathscr{C}$ is a G_δ set. In particular, if $\mathscr{C}$ is dense, it is a dense G_δ.

In the following, we suppose

$$H(\omega) = H_0 + V_\omega \tag{2.1}$$

where H_0 is self-adjoint with domain $\mathscr{D}$ and the V_ω form a strongly continuous family of bounded self-adjoint operators.

Proposition 2.2. Suppose $\mathscr{D}(A) \cap \mathscr{D}$ is dense and

(i) $\forall \omega \in \mathbb{R}^n$, $\forall t \in \mathbb{R}$, $U_t(\omega)\, \mathscr{D}(A) \subset \mathscr{D}(A)$;

(ii) $\forall \omega \in \mathbb{R}^n$, $[H(\omega), A]$ is relatively H_0 bounded and the map $\omega \in \mathbb{R}^n \to [H(\omega), A](H_0 + i)^{-1} \in \mathscr{L}(\mathscr{H})$ is strongly continuous.

Suppose in addition that for some $\psi \in \mathscr{D}(A) \cap \mathscr{D}$ there exists a $\beta_0 > 0$ so that for all ω in a dense subset $\mathscr{R}_\psi$ of $\mathbb{R}^n$

$$\beta_0 = \sup \left\{ \alpha > 0 \,\middle|\, \int_1^\infty \frac{dt}{t^{1+\alpha}} \overline{\langle U_t(\omega)\,\psi, A^*AU_t(\omega)\,\psi \rangle} = \infty \right\} \tag{2.2}$$

Then for all ω in a dense G_δ set $\mathscr{S}_\psi$ one has:

$$\beta_0 \leqslant \sup \left\{ \alpha > 0 \,\middle|\, \int_1^\infty \frac{dt}{t^{1+\alpha}} \overline{\langle U_t(\omega)\,\psi, A^*AU_t(\omega)\,\psi \rangle} = \infty \right\} \tag{2.3}$$

Remark. (i) Proposition 2.1 is Theorem 1.1 in [Si]. We have preferred to give an independent proof based on the use of the Wiener theorem, since the same type of argument serves to show Propositions 2.2–2.4. (ii) As in [Si], $\mathbb{R}^n$ can be replaced by a complete metric space. (iii) The time-average appearing in (2.2) and (2.3) can be omitted for the abstract argument, but often appears in applications.

Proof of Proposition 2.1. Let ψ_i, $i \in \mathbb{N}$ be an orthonormal basis of $\mathcal{H}$. Consider for each $i \in \mathbb{N}$, $T \in [1, \infty[$ the continuous functions

$$R^i_T\colon \omega \in \mathbb{R}^n \to \left[\frac{1}{T}\int_0^T |\langle \psi_i, U_t(\omega)\,\psi_i\rangle|^2\,dt\right] \in \mathbb{R}$$

Then Wiener's theorem immediately implies that

$$\mathscr{C} = \bigcap_{i,n \in \mathbb{N}} \bigcup_{T \in \mathbb{N}} \left\{\omega \,\middle|\, R^i_T(\omega) < \frac{1}{n}\right\}$$

This proves the proposition. ▮

Proof of Proposition 2.2. We first show that the maps

$$\omega \in \mathbb{R}^n \to \langle U_t(\omega)\,\psi, A^*A\,U_t(\omega)\,\psi\rangle \in \mathbb{R} \tag{2.4}$$

are continuous for all $\psi \in \mathscr{D}(A)$. For that purpose, compute for $\psi \in \mathscr{D}(A) \cap \mathscr{D}$

$$\begin{aligned}(A\,U_t(\omega) &- A\,U_t(\omega'))\,\psi\\ &= [A, U_t(\omega)]\,\psi - [A, U_t(\omega')]\,\psi + (U_t(\omega) - U_t(\omega')\,A\psi\end{aligned}$$

and

$$[A, U_t(\omega)] = U_t(\omega)\, i \int_0^t ds\, U_{-s}(\omega)[H(\omega), A]\, U_s(\omega)$$

Continuity of (2.4) now follows easily. For all $\varepsilon > 0$ and for all $\omega \in \mathscr{R}_\psi$, (2.2) implies

$$\sup_T \int_1^T \frac{dt}{t^{1+\beta_0-\varepsilon}}\, \overline{\langle U_t(\omega)\,\psi, A^*A\,U_t(\omega)\,\psi\rangle} = \infty \tag{2.5}$$

It is a standard corollary of the Baire cathegory theorem ([S], Corollaire 4, p. 325) that (2.5) continues to hold for all ω in a dense G_δ set $\mathscr{S}^\varepsilon_\psi$, containing $\mathscr{R}_\psi$, and hence

$$\int_1^\infty \frac{dt}{t^{1+\beta_0-\varepsilon}}\, \overline{\langle U_t(\omega)\,\psi, A^*A\,U_t(\omega)\,\psi\rangle} = \infty, \qquad \forall \omega \in \mathscr{S}^\varepsilon_\psi \tag{2.6}$$

Define $\mathscr{S}_\psi = \bigcap_{n \in \mathbb{N}^*} \mathscr{S}^{1/n}_\psi$. Then (2.6) implies that for all $\omega \in \mathscr{S}_\psi$ (2.3) holds. ▮

Analogous results hold when dealing with iterates of a unitary operator $U(\omega)$. We state the results without proofs, which are completely analogous.

Proposition 2.3. Let $\omega \in \mathbb{R}^n \to U(\omega)$ be a strongly continuous map. If the pure point spectrum of $U(\omega)$ is empty on a dense set of ω, then this remains true on a dense G_δ set.

Proposition 2.4. Suppose

(i) $\omega \in \mathbb{R}^n \to U(\omega) \in \mathscr{U}(\mathscr{H})$ is strongly continuous;

(ii) $\forall \omega \in \mathbb{R}$, $U(\omega)\, \mathscr{D}(A) \subset \mathscr{D}(A)$;

(iii) $\forall \omega \in \mathbb{R}^n$, $[U(\omega), A]$ is bounded and the map $\omega \in \mathbb{R}^n \to [U(\omega), A] \in \mathscr{L}(\mathscr{H})$ is strongly continuous.

Suppose in addition that for some $\psi \in \mathscr{D}(A)$ there exists a $\beta_0 > 0$ so that

$$\beta_0 = \sup \left\{ \alpha > 0 \;\middle|\; \sum_{m=1}^{\infty} \frac{1}{m^{1+\alpha}} \overline{\langle U(\omega)^m \psi, A^* A U(\omega)^m \psi \rangle} = \infty \right\} \tag{2.7}$$

for all ω in a dense subset $\mathscr{R}_\psi$ of $\mathbb{R}^n$. Then for all ω in a dense G_δ set $\mathscr{S}_\psi$

$$\beta_0 \leqslant \sup \left\{ \alpha > 0 \;\middle|\; \sum_{m=1}^{\infty} \frac{1}{m^{1+\alpha}} \overline{\langle U(\omega)^m \psi, A^* A U(\omega)^m \psi \rangle} = \infty \right\} \tag{2.8}$$

3. APPLICATIONS

3.1. The Kicked Rotor

It is hard to get rigorous results on the dynamics or on the dynamical or spectral stability for the kicked rotor [IS]. For the quantum model, with Floquet operator

$$U_V(\omega) = \exp - i\omega \frac{P^2}{2} \exp - iV(X)$$

acting on $L^2(\mathbf{S}^1)$ the only rigorous result we know of is due to Casati-Guarneri [CG]. In the following theorem we give both an improvement on this result and a simplified proof of it.

Theorem 3.1. Suppose $V \in L^2(\mathbf{S}^1)$, $V' \in L^\infty(\mathbf{S}^1)$ and that $\forall \pi\omega \in \mathbb{Q}$, the spectrum of $U_V(\omega)$ is purely absolutely continuous. Then there exists a dense G_δ subset $\mathscr{S}$ of $\mathbb{R}$ so that

(i) $\forall \omega \in \mathscr{S}$, $\sigma_{pp}(U_V(\omega)) = \varnothing$;

(ii) $\forall \omega \in \mathscr{S}$, for all momentum eigenstates $\psi_\ell(x) = \exp i2\pi\ell x$, $\ell \in \mathbb{Z}$,

$$\sup\left\{\alpha > 0 \,\middle|\, \sum_{m=1}^{\infty} \frac{1}{m^{1+\alpha}} \overline{\langle \psi_\ell, U_V(\omega)^{-m} P^2 U_V(\omega)^m \psi_\ell \rangle} = \infty \right\} = 2 \tag{3.1}$$

i.e., the mean growth exponent of the time-averaged kinetic energy is 2.

The proof of this result is given below. Let us first point out that the hypothesis on $\sigma(U_V(\omega))$ for $\pi\omega \in \mathbb{Q}$ is proven to hold for a generic set of V in [CG]. Note that part (i) of Theorem 3.1 is Theorem 2 of [CG]. Actually, the statement of Theorem 2 in [CG] is weaker than the one given here, but Theorem 3.1(i) immediately follows from their proof. Part (ii) shows that for ω belonging to $\mathscr{S}$, the time-averaged kinetic energy displays "almost ballistic peaks" at sufficiently many time scales to force the mean growth exponent α_0 to equal 2. We have no precise information on how frequent these time scales are, but expect them to be extremely rare, and hence difficult to detect numerically. Since the proof of Theorem 3.1 is based on the abstract nonsense of the previous section, we have no handle on the set $\mathscr{S}$ either. It should nevertheless be thought of as a set of Liouville numbers. The main open problem in the kicked rotor is to show that for ω a quadratic irrational and for sufficiently high coupling constant dynamical localization occurs: we unfortunately have nothing to say on that.

Proof of Theorem 3.1. It will be sufficient to check

$$U_V(\omega) = \exp - i\omega \frac{P^2}{2} \exp - iV(X)$$

satisfies the conditions of Propositions 2.3 and 2.4 with $A = P$. That $\omega \to U_V(\omega)$ is strongly continuous is immediate and since

$$PU_V(\omega)\,\psi = U_V(\omega)\,P\psi - U_V(\omega)\,V'\psi$$

hypotheses (ii)–(iii) of Proposition 2.4 follow as well. Hence part (i) of Theorem 3.1 follows from Proposition 2.3. As for part (ii), since for all $\pi\omega \in \mathbb{Q}$, $\sigma_s(U_V(\omega)) = \varnothing$, the results of Last and Guarneri [G, L] show that $\forall \pi\omega \in \mathbb{Q}$ and $\forall \psi_\ell$ $\exists C_\ell(\omega)$ so that

$$\overline{\langle U_V(\omega)^m \psi_\ell, P^2 U_V(\omega)^m \psi_\ell \rangle} \geqslant C_\ell(\omega)\, m^2 \tag{3.2}$$

For all $\pi\omega \in \mathbb{Q}$, the mean growth exponent is therefore equal to 2. The result then follows from Proposition 2.4, provided we show that, for all ω, the mean growth exponent is smaller or equal than two. This is obvious upon remarking that

$$U_V^{-m} P^2 U_V^m = \left[P - \sum_{k=0}^{k=m-1} U_V^{-k} V' U_V^k \right]^2 \quad \blacksquare$$

Note that there are other models, such as the kicked Harper model, where resonances occur, and the results can be adapted to such cases as well.

3.2. Quasiperiodic Potentials

Let Ω be an $n+1$ by n matrix and W a continuous $\mathbb{Z}^{n+1}$-periodic real-valued function on $\mathbb{R}^{n+1}$. Define, for $x \in \mathbb{R}^n$,

$$V_\Omega(x) = W(\Omega x)$$

Note that, if Ω has only rational entries, there is a sublattice of $\mathbb{Z}^n$ over which V_Ω is periodic. As a result, the Schrödinger operator

$$H_\Omega = -\Delta + V_\Omega$$

on $\mathscr{D}(H_\Omega) = \mathscr{D}(-\Delta)$ has purely absolutely continuous spectrum. For $\psi \in \mathscr{D}(X)$ one then has that $\langle \psi_t, X^2 \psi_t \rangle^2 \sim t^2$ (see [AK] for a detailed proof of this folk theorem). Identifying Ω with an element of $\mathbb{R}^{n(n+1)}$, it is easy to see that the hypotheses of Proposition 2.1 and Proposition 2.2 are satisfied by H_Ω, taking $A = X$ and $\psi \in \mathscr{D}(X)$. As a result, there is a G_δ dense set of ω so that the mean growth exponent of $\langle \psi_t, X^2 \psi_t \rangle$ is 2 and, in particular, for all strictly monotonic functions $F(T)$, $F(T) \to \infty$, $\exists C > 0$, $T_k \to \infty$, so that

$$\langle \psi, e^{iH_\Omega T_k} X^2 e^{-iH_\Omega T_k} \psi \rangle \geqslant C \frac{T_k^2}{F(T_k)} \tag{3.3}$$

This is of course easily adapted to discrete Schrödinger operators and holds for the time-averaged quantities as well. In fact, combining the above with Theorem 7.1 in [L], one gets the following result on the almost Mathieu equation.

Theorem 3.2. Let $H_{\lambda,\theta}(\omega) = -\Delta + \lambda \cos 2\pi(\omega n + \theta)$ on $\ell^2(\mathbb{Z})$, with $|\lambda| > 2$ and $\theta \in \mathbb{R}$. Then there exists a dense G_δ set of ω so that

(i) All spectral measures of $H_{\lambda,\theta}(\omega)$ are zero Hausdorff dimensional;

(ii) For all ψ in a dense subset of $\ell^2(\mathbb{Z})$ contained in $\mathscr{D}(X)$, (3.3) holds and the mean growth exponent of $\langle \psi_t, X^2 \psi_t \rangle$ equals 2.

Here (i) is exactly Theorem 7.1 of [L], while (ii) improves Theorem 7.2 of [L]. It follows immediately upon taking intersections of dense G_δ sets.

4. THE KICKED LINEAR ROTOR

4.1. The Model

Consider a particle moving on a circle $\mathbf{S}^1$, having therefore the cylinder $M = T^*\mathbf{S}^1 = \mathbf{S}^1 \times \mathbb{R}$ as phase space. The Hamiltonian of the kicked linear rotor [G, Be, B, Ho, O] is

$$H(x, p, t) = \omega p + \sum_{n \in \mathbb{Z}} \delta(t-n)\, V(x) \tag{4.1}$$

where $\omega \in \mathbb{R}$ is a fixed rotation number. The corresponding Floquet map is

$$\Phi_V(x_0, p_0) = (x_1 = x_0 + \omega,\ p_1 = p_0 + f(x_0)) \tag{4.2}$$

where $f = -V'$. Standard canonical quantization of the Hamiltonian in (4.1) leads to the following Floquet operator on $L^2(\mathbf{S}^1, dx)$:

$$U_V(\omega) = \exp - i\omega P \exp - iV(X) \tag{4.3}$$

To have some explicit examples in mind, we define, for $k \in \mathbb{N}\backslash\{0\}$,

$$Q_k(x) = \frac{1}{(2\pi i)^k} \sum_{n \in \mathbb{Z}^*} \frac{1}{n^k} \exp i2\pi n x$$

Then $Q_1(x) = \frac{1}{2} - x$, $0 < x < 1$, and for all $k > 1$, Q_k is the unique kth order polynomial satisfying $Q_k' = Q_{k-1}$, $\int_0^1 Q_k(x)\, dx = 0$. Note that $Q_k^{(i)}(0) = Q_k^{(i)}(1)$ for all $0 \leqslant i \leqslant k-2$. Hence $Q_k \in C^{(k-2)}(\mathbf{S}^1)$. For example, $Q_2(x) = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{12}$ and $Q_3(x) = \frac{1}{12}x(x-1)(-2x+1)$.

In order to study the asymptotic behaviour of the momentum variable in the kicked linear rotor, first note that iterating (4.2) yields

$$\Phi_V^m(x_0, p_0) = (x_m = x_0 + m\omega,\ p_m = p_0 + S(m, \omega)\, f(x_0)) \tag{4.4}$$

where

$$S(m,\omega)\,f(x_0)=\sum_{j=0}^{m-1} f(x_0+j\omega) \tag{4.5}$$

Let us point out some immediate and simple features of this model.

First, if $\omega\in\mathbb{Q}$, then typically $p_m \sim m$ as is easily seen from (4.4)–(4.5) and the observation that the motion is then periodic in the x-variable. Indeed, if $\omega=r/s$, with r, s relatively prime integers, then, for all $\ell\in\mathbb{N}$

$$\Phi_V^{\ell s}(x_0,p_0)=\left(x_{\ell s}=x_0,\ p_{\ell s}=p_0+\ell S\left(s,\frac{r}{s}\right)\,f(x_0)\right)$$

So $p_{\ell s}\sim\ell$ iff $S(s,(r/s))\,f(x_0)\neq 0$.

On the other hand, if $\omega\in\mathbb{R}\backslash\mathbb{Q}$, then the motion is ergodic in the x-variable and hence one immediately obtains that $\lim_{m\to\infty}(1/m)\ S(m,\omega)\,f(x_0)=0$ so that

$$\lim_{m\to\infty}\frac{p_m}{m}=0 \tag{4.6}$$

In short, when ω is rational, the motion is ballistic on the cylinder, whereas it is subballistic if ω is irrational.

Note however that, even though $p_m=o(m)$ it is still conceivable that $\sup_m |p_m|=\infty$. Moreover it is well known from the Denjoy-Koksma inequality that $\liminf_m |p_m|<C<\infty$ if $f=-V'\in C^1(\mathbf{S}^1)$. Hence the classical dynamics would in such a case correspond to the point (x_m, p_m) wandering all over the cylinder, up and down the p axis, leaving every bounded set at some times and returning close to the origin at some other times. This behaviour is thought of as unusual since it can not occur in systems with time-independent Hamiltonians [RS III]; one expects it to be reflected in the quantum model through the presence of singular continuous spectrum, a fact we shall prove below.

The asymptotics for the classical dynamics of the kicked linear rotor was studied in detail in [DBF]. We shall show here that the analysis carries over to the quantum model and study the spectral and dynamical (in)stability of the model.

Before turning to this task, we show the quantum equivalent of (4.6). We need a preliminary technical lemma. Let $\mathscr{D}(P)$ denote the domain of P, i.e.,

$$\mathscr{D}(P)=\left\{\psi=\sum c_n \exp i2\pi nx\in L^2(\mathbf{S}^1,dx);\ \sum n^2\,|c_n|^2<\infty\right\}$$

Lemma 4.1. If $\psi \in \mathscr{D}(P)$ and if $V, V' \in L^2(\mathbf{S}^1)$, then $U_V^m \psi \in \mathscr{D}(P)$ $\forall m \in \mathbb{N}$. Moreover,

$$\langle \psi, U_V^{-m} P U_V^m \psi \rangle = \langle \psi, S(m, \omega)\, f(X)\, \psi \rangle + \langle \psi, P\psi \rangle \tag{4.7}$$

and

$$\| P U_V^m \psi \|^2 = \langle U_V^m \psi, P^2 U_V^m \psi \rangle = \| P \exp - iS(m, \omega)\, V(X)\, \psi \|^2 \tag{4.8}$$

where $\|\cdot\|$ denotes the L^2-norm.

Proof. First we compute

$$\begin{aligned} U_V^m &= \exp - i\omega P m \exp - i \sum_{k=0}^{m-1} V(X + k\omega) \\ &= \exp - i \sum_{k=1}^{m} V(X - k\omega) \exp - i\omega P m \end{aligned} \tag{4.9}$$

Now, if $\psi \in \mathscr{D}(P)$, then $\psi \in L^\infty(\mathbf{S}^1)$, so $f(x - k\omega)\, \psi(x - m\omega) \in L^2(\mathbf{S}^1, dx)$ since $f \in L^2(\mathbf{S}^1, dx)$. In conclusion

$$P(U_V^m \psi)(x) = \sum_{k=1}^{m} f(x - k\omega)(U_V^m \psi)(x) + (U_V^m(P\psi))(x)$$

belongs to $L^2(\mathbf{S}^1, dx)$; (4.7) follows immediately from this. Using the first relation in (4.9), (4.8) follows as well. ∎

Proposition 4.2. If $\psi \in \mathscr{D}(P)$, if $V, V' \in L^2(\mathbf{S}^1)$, and if ω is irrational one has, as $m \to \infty$,

$$\frac{1}{m} \langle \psi, U_V^{-m} P U_V^m \psi \rangle \to 0 \tag{4.10}$$

and

$$\frac{1}{m^2} \| P U_V^m \psi \|^2 \to 0 \tag{4.11}$$

Proof. Since $\int_{\mathbf{S}^1} f(x)\, dx = 0$ and $f \in L^2(\mathbf{S}^1, dx)$ it follows from the Von Neumann mean ergodic theorem that $m^{-2} \| S(m, \omega) f \|^2 \xrightarrow{m \to \infty} 0$. Hence, using (4.8), one has (4.11) since $\psi \in L^\infty(\mathbf{S}^1)$. The proof of the first statement is similar. ∎

Clearly (4.10)–(4.11) are the quantum analogs of (4.6). For rational ω, one sees from (4.7) that, as in the classical case, $\langle \psi, U_V^{-m} P U_V^m \psi \rangle \sim m$ for *typical potentials* V; we shall say that a potential V is a *typical potential* provided the function $S(s, (r/s))\, V$ is not constant on a set of positive measure for any r, s as above. Note that it follows from our discussion above that this is equivalent to requiring that $p_m \sim m$ for all rational ω and for almost all initial values x_0. Putting an appropriate topology on the set of all potentials one can show that the typical potentials form a dense G_δ set, but we shall not be interested in that any further. It is perhaps more interesting to remark that the potentials Q_k introduced above are typical. Indeed, it is easy to compute

$$S\left(s, \frac{r}{s}\right) Q_k(x) = s^{1-k} Q_k(sx)$$

from which the conclusion follows.

4.2. Spectral (In)Stability

We are now ready to settle the spectral stability question for the kicked linear rotor. The result is summarized in the following theorem.

Theorem 4.3. Let $U_V(\omega) = \exp - i\omega P \exp - iV(X)$ with $V, V' \in L^2(\mathbf{S}^1)$, i.e., $V \in H^1(\mathbf{S}^1)$. Then

(i) For typical V and for all $\omega \in \mathbb{Q}$, $\sigma_{pp}(U_V(\omega)) = \emptyset = \sigma_{sc}(U_V(\omega))$; i.e., $L^2(\mathbf{S}^1) = \mathscr{H}_{ac}(U_V(\omega))$;

(ii) If $\omega \in \mathbb{R} \backslash \mathbb{Q}$, then $\sigma_{ac}(U_V(\omega)) = \emptyset$ and in addition either $L^2(\mathbf{S}^1) = \mathscr{H}_{sc}(U_V(\omega))$ or $L^2(\mathbf{S}^1) = \mathscr{H}_{pp}(U_V(\omega))$;

(iii) For typical V, there exists a dense G_δ set of ω so that $L^2(\mathbf{S}^1) = \mathscr{H}_{sc}(U_V(\omega))$ (i.e., $\sigma_{ac}(U_V(\omega)) = \emptyset = \sigma_{pp}(U_V(\omega))$;

(iv) If $V \in H^s(\mathbf{S}^1)$, $s > 1$, then $L^2(\mathbf{S}^1) = \mathscr{H}_{pp}(U_V(\omega))$ (i.e., $\sigma_{ac}(U_V(\omega)) = \emptyset = \sigma_{sc}(U_V(\omega))$) for Lebesgue almost all ω.

Proof. Part (i) is the spectral pendant of the observation made above that for rational ω, and for typical V, the motion is ballistic: $p_m \sim m$. If $\omega = r/s$ with r and s relatively prime integers, then it is easy to see that the spectrum of $U_V(\omega)$ is organized in bands, and has no singular continuous part [IS, Be]. To see that, for typical V, it has no eigenvalues either, we proceed as follows. First, (4.9) implies

$$U_V(\omega)^s = \exp - i \sum_{k=0}^{s-1} V\left(X + k\frac{r}{s}\right)$$

Consequently, it is clear that, provided the function $S(s, (r/s))\, V$ is not constant on a set of positive measure, the spectrum of $U_V(\omega)^s$ is purely absolutely continuous. In that case, the same holds for the spectrum of $U_V(\omega)$ itself.

More interesting is the case where $\omega \in \mathbb{R}\backslash\mathbb{Q}$. To prove (ii), recall that the aforementioned result of Guarneri and Last, and more precisely Theorem 6.1 in [L] implies that, if the spectral measure of ψ contains an absolutely continuous component, then necessarily

$$\overline{\langle \psi, U_V^{-m} P^2 U_V^m \psi \rangle} \geqslant Cm^2, \qquad \forall m \in \mathbb{Z}$$

for some $C > 0$. This being incompatible with (4.11), we conclude that $\sigma_{ac}(U_V(\omega))$ is empty, proving the first statement in (ii); (iii) now follows immediately from this, Proposition 2.3, and (i).

The second part of (ii) is easily proven as follows. Suppose $\lambda \in \sigma_{pp}(U_V(\omega))$ and let ψ be the corresponding eigenfunction, then

$$U_V \psi(x) = \exp - iV(x - \omega)\, \psi(x - \omega) = \lambda \psi(x) \tag{4.12}$$

Taking absolute values on both sides yields

$$|\psi|\,(x - \omega) = |\psi|\,(x), \qquad \forall x \in [0, 1[$$

Since ω is irrational, this implies $|\psi|\,(x) \equiv \mathrm{cst}$, so that $\psi(x) = \exp - iW(x)$ for some W. Reinserting this into (4.12), one has

$$\exp - iV(x - \omega) = \lambda \exp - iW(x) \exp iW(x - \omega)$$

and hence

$$U_V(\omega) = \lambda \exp - iW(X) \exp - i\omega P \exp iW(X) \tag{4.13}$$

So $U_V(\omega)$ is unitarily equivalent to $\lambda \exp - i\omega P$, which of course has pure point spectrum. In conclusion, if $\omega \in \mathbb{R}\backslash\mathbb{Q}$, we know that either $\sigma_c(U_V(\omega)) = \varnothing$, or $\sigma_{pp}(U_V(\omega)) = \varnothing$; the spectrum is therefore either purely singular continuous or exclusively pure point. We have already shown that the first case occurs on a dense G_δ set. We now show the second case occurs as well.

One expects the spectrum to be pure point for ω poorly approximated by the rationals. To prove this, first remark that it follows from (4.13) that it suffices to solve

$$V(x) = W(x + \omega) - W(x)$$

for $W \in L^2(\mathbf{S}^1)$. Assuming $V \in H^s(\mathbf{S}^1)$, $s > 1$ and that $q_{k+1} = O(q_k^{1+\gamma})$ with $1 + \gamma < s$ such a solution exists as is easily seen upon Fourier transforming the equation and solving for the Fourier coefficients of W. It is then easy to see that $U_V(\omega)$ is of the form (4.13) with $\lambda = 1$. Here the q_k are the denominators of the continuous fraction approximants of ω. Since the condition $q_{k+1} = O(q_k^{1+\gamma})$ holds Lebesgue a.e. (Theorem 32 in [K]), (iv) follows. ▮

Remark. A version of part (i) of this result is proven in [Ho, Be]. Absolute continuity of the spectrum implies [G, L] that $\forall \psi \in \mathscr{D}(P)$,

$$\overline{\langle \psi, U_V^{-m} P^2 U_V^m \psi \rangle} \geqslant Cm^2, \qquad \forall m \in \mathbb{Z}$$

a fact we will prove more directly below. Part (ii) is an improvement over [Be], where the result is proven on the one hand for all irrational ω under the hypothesis that $V \in C^1$, $\int_{\mathbf{S}^1} |V'|(x)\,dx < 2\pi$ and on the other hand, for almost all $\omega \in \mathbb{R} \backslash \mathbb{Q}$, provided $V \in C^2(\mathbf{S}^1)$. Our result here shows that $V' \in L^2(\mathbf{S}^1)$ suffices and that the condition on the total variation of V can be removed. Note that some smoothness of V is at any rate needed: it is mentioned in [Be] and easily confirmed that for $V(x) = 2\pi x$, $x \in]0, 1[$, the spectrum of U_V is a.c. even for irrational ω. Note however that, viewed as a function on $\mathbf{S}^1$, this potential does not have an L^2 derivative because of the jump discontinuity. For typical $V \in C^2(\mathbf{S}^1)$, $\int |V'|\,dx < 2\pi$, (iii) is proven in [O] using the results of [CG] and [Be].

4.3. Dynamical (In)Stability

We are interested in the behaviour of $\overline{\langle U_V^m \psi, P^2 U_V^m \psi \rangle}$. First remark that the results of Section 2 immediately imply that, generically in ω, the mean growth exponent of this quantity equals 2. To get sharper estimates, we now use the results of [DBF]. If the Fourier coefficients v_n of V satisfy $v_n \sim |n|^{-(\nu+1)}$ for some $\nu > 1/2$, it is an immediate consequence of Theorem 1.1(i) of [DBF] that for a class $\mathscr{R}_\infty$ of explicitly described Liouville ω and for all $\varepsilon > 0$, there exists a constant C_ε so that

$$\overline{\langle \psi_\ell, U_V(\omega)^{-m} P^2 U_V(\omega)^m \psi_\ell \rangle} \geqslant C_\varepsilon m^{(2/1+\nu)-\varepsilon}, \qquad \forall m \qquad (4.14)$$

where $\psi_\ell(x) = \exp i2\pi\ell x$. Here $\omega \in \mathscr{R}_\infty$ if and only if the denominators q_k of its continued fraction expansion satisfy

$$\forall \gamma > 0, \quad \exists R_\gamma > 0 \qquad \text{so that} \quad q_{k+1} > \frac{q_k^{1+\gamma}}{R_\gamma}$$

Moreover, Theorem 1.2(ii) in [DBF] implies that, for the same ω,

$$\frac{2}{1+\nu} \leqslant \alpha_{-} = \liminf_{m \to \infty} \frac{\log \overline{\langle \psi_\ell, U_V(\omega)^{-m} P^2 U_V(\omega)^m \psi_\ell \rangle}}{\log m} \leqslant \frac{2}{1/2+\nu} \tag{4.15}$$

Using (1.5) this implies that $\dim_H \mu_{\psi_\ell} \leqslant (1/(1/2)+\nu)$. At the same time, and still for the same ω, Theorem 1.1 in [DBF] yields

$$\alpha_{+} = \limsup_{m \to \infty} \frac{\log \overline{\langle \psi_\ell, U_V(\omega)^{-m} P^2 U_V(\omega)^m \psi_\ell \rangle}}{\log m} = 2 \tag{4.16}$$

Summarizing the result loosely, we see that $\overline{\langle \psi_\ell, U_V(\omega)^{-m} P^2 U_V(\omega)^m \psi_\ell \rangle}$ oscillates quite a bit, between m^2 and $m^{(2/(\nu+1/2))}$, staying always at least as high as $m^{2/(\nu+1)}$.

Note again that the upper bound on the Hausdorff dimension of the spectral measure μ_{ψ_ℓ} does not exclude the existence of almost ballistic peaks, a phenomenon already observed in [L] and in Theorem 3.2 above.

If $V \in C^\infty$, then Theorem 1.2 (ii) of [DBF] shows that, for all $\omega \in \mathbb{R} \backslash \mathbb{Q}$, $\alpha_{-} = 0$, implying, via (1.5), that the Hausdorff dimension of μ_ψ vanishes. So in this case, the Hausdorff dimension takes on only two values: it is equal to 1 if ω is rational and equal to 0 otherwise. At the same time, the results of Section 2 tell us that for generic values of ω, $\alpha_0 = 2 = \alpha_{+}$. This shows in an even stronger way than the example of Section 3.2 that no upper bounds on $\overline{\langle \psi, U_V(\omega)^{-m} P^2 U_V(\omega)^m \psi \rangle}$ can be expected in terms of information on the fractal dimension of the spectral measures.

These results show that $\overline{\langle \psi, U_V(\omega)^{-m} P^2 U_V(\omega)^m \psi \rangle}$ tends to fluctuate enormously. It is legitimate to speculate that this is a feature common to the solution of Schrödinger equations with propagators having "unusual" spectrum. Note however that detecting such fluctuations numerically might be an impossible task: the times at which they are proven to occur (see [DBF]) behave like q_k, where the q_k are the denominators of the convergents of the continued fraction expansion of ω, which grow extremely fast for the ω considered.

To end this section, we should point out that the results in [DBF] actually concern the asymptotic behaviour of $\langle \overline{p_m} \rangle^2$, where

$$\langle \bar{p}_m \rangle^2 = \int_{S^1} |\bar{p}_m(x_0)|^2 \, dx_0$$

This is clearly determined by the asymptotic behaviour of the L^2-norm of the time-average of $S(m, \omega) f$. It turns out that this quantity has the same asymptotic behaviour as $\overline{\langle \psi_\ell, U_V(\omega)^{-m} P^2 U_V(\omega)^m \psi_\ell \rangle}$, as we now briefly show. First compute, using (4.8)

$$[\|S(m,\omega) f(X) \psi\| - \|P\psi\|]^2 \leqslant \langle U_V^m \psi, P^2 U_V^m \psi \rangle \leqslant [\|S(m,\omega) f(X) \psi\| + \|P\psi\|]^2 \tag{4.17}$$

Averaging over m and a short computation then yield

$$\left[\left(\frac{1}{m} \sum_{k=1}^{m} \|S(k,\omega) f(X) \psi\|^2\right)^{1/2} - \|P\psi\|\right]^2 \leqslant \frac{1}{m} \sum_{k=1}^{m} \langle U_V^k \psi, P^2 U_V^k \psi \rangle \leqslant \left[\left(\frac{1}{m} \sum_{k=1}^{m} \|S(k,\omega) f(X) \psi\|^2\right)^{1/2} + \|P\psi\|\right]^2$$

from which it is clear that the asymptotic behaviour of $\overline{\langle U_V^m \psi, P^2 U_V^m \psi \rangle}$ is determined by the one of $\overline{\|S(m,\omega) f(X) \psi\|^2}$. Assuming that $|\psi|(x) = 1$, $\forall x$, which is in particular true when ψ is a momentum eigenstate $\psi_\ell(x)$, one simply has $\|S(m,\omega) f(X) \psi_\ell\| = \|S(m,\omega) f(X)\|$ so that the averaged kinetic energy is then determined by the time-averaged L^2-norm of $S(m,\omega) f(X)$. It turns out that this has the same asymptotic behaviour as the L^2-norm of the time-average of $S(m,\omega) f$. To see this, note that a simple computation shows that

$$\left\| \frac{1}{M} \sum_{m=1}^{M} S(m,\omega) f(X) \right\|^2 = \sum_{n \in \mathbb{Z}} |f_n|^2 G_M(n\omega)$$

with

$$G_M(x) = \frac{1}{4\pi \sin^2 \pi x} \left| 1 - \frac{\sin \pi M x}{M \sin \pi x} \exp i\pi M x \right|^2$$

whereas

$$\frac{1}{M} \sum_{m=1}^{M} \|S(m,\omega) f(X)\|^2 = \sum_{n \in \mathbb{Z}} |f_n|^2 H_M(n\omega)$$

with

$$H_M(x) = \frac{1}{2 \sin^2 \pi x} \left[1 - \frac{\cos \pi x(M+1) \sin \pi x M}{M \sin \pi x} \right]$$

It then suffices to notice that the proofs of [DBF] are entirely based on the following estimates for $G_M(x)$ (Lemma 2.1 in [DBF])

$$c/\sin^2 \pi x \leqslant G_M(x) \leqslant C/\sin^2 \pi x, \qquad \forall 1/M \leqslant x \leqslant 1/2$$

and

$$cM^2 \leqslant G_M(x) \leqslant CM^2 \qquad \forall 0 \leqslant x \leqslant 1/M$$

with the upper bound holding for all $x \in \mathbb{R}$. The same inequalities hold for $H_M(x)$ as well, as is easily checked. It follows that the quantum and classical dynamics in this model are essentially the same. This is not too surprising if one compares $p_m = p_0 + S(m, \omega)\, f(x_0)$ to (4.7), which can be rewritten suggestively as (see also [O])

$$P_m \equiv U_V^{-m} P_0 U_V^m = P_0 + S(m, \omega)\, f(X_0)$$

5. A ONE-BAND TIGHT BINDING MODEL IN A TIME-DEPENDENT ELECTRIC FIELD

In this section we turn our attention to the one-dimensional tight-binding model with time-dependent electric field $E(t)$ given by

$$[H(t)\,\psi]_n = \sum_{k \in \mathbb{Z}} a_{n-k} \psi_k + E(t)\, n\psi_n \tag{5.1}$$

Here the off-diagonal matrix elements a_n of the Hamiltonian are assumed to belong to $\ell^2(\mathbb{Z})$ and $a_n = a_{-n}$. The corresponding time-dependent Schrödinger equation

$$i\, \partial_t \psi_t = H(t)\, \psi_t \tag{5.2}$$

can be solved explicitly in several ways [DK]. We proceed as follows. Fourier transforming of (5.2) yields

$$i\, \partial_t \hat{\psi}_t = \hat{H}(t)\, \hat{\psi}_t$$

where, for $x \in [0, 1[\equiv \mathbf{S}^1$,

$$\hat{\psi}(x) = \sum_n \psi_n \exp i2\pi nx \in L^2(\mathbf{S}^1, dx)$$

$$\hat{a}(x) = \sum_n a_n \exp i2\pi nx \in L^2(\mathbf{S}^1, dx)$$

$$\hat{H}(t) = \frac{E(t)}{2\pi} P + \hat{a}(x), \qquad P = \frac{1}{i} \frac{d}{dx}$$

We chose to use the unusual notation "x" for the quasi-momentum in order to bring the analogy with the previous section out more clearly, as follows. Solving the Schrödinger equation yields

$$\hat{\psi}_t = U_t \hat{\psi}_0 \tag{5.3}$$

where

$$U_t = \exp - iG(t)\, P \exp - iW_t(X) \tag{5.4}$$

and

$$G(t) = \frac{1}{2\pi} \int_0^t E(s)\, ds; \qquad W_t(x) = \int_0^t \hat{a}(x + G(s))\, ds \tag{5.5}$$

We now concentrate on smooth periodic field amplitudes $E(t)$ of period T for which we write

$$E(t+T) = E(t); \qquad E(t) = E_0 + e(t); \qquad e(t) = \sum_{n \in \mathbb{Z}^*} e_n \exp i \frac{2\pi}{T} nt \tag{5.6}$$

The Floquet operator U_T is then given by

$$U_T = \exp - i\tau P \exp - iW_T(x), \qquad \tau = \frac{E_0 T}{2\pi} \tag{5.7}$$

The analogy with the previous section, and in particular with (4.3) is now completely clear. We shall be interested in the asymptotic behaviour of

$$\langle \psi_t, N^2 \psi_t \rangle = \sum_{n \in \mathbb{Z}} n^2 |\psi_{tn}|^2 = \frac{1}{(2\pi)^2} \langle \hat{\psi}_t, P^2 \hat{\psi}_t \rangle \tag{5.8}$$

for $\psi_0 \in \mathscr{D}(N)$. For this expression to make sense, we need that $\psi_t \in \mathscr{D}(N)$. A sufficient condition to ensure that $\mathscr{D}(N)$ is invariant under U_t is given by Lemma 4.1: $na_n \in \ell^2(\mathbb{Z})$.

Note first that, if $E_0 = 0$, it is well known that $\langle \psi_t, N^2 \psi_t \rangle \sim t^2$, except if $e(t) = E_1 \cos(2\pi/T)\, t$, $a_n = \delta_{1,n} + \delta_{-1,n}$ and with very special choices of T and E_1 [DK]. If, on the other hand, $e(t) = 0$, one readily sees that $\sup_t \langle \psi_t, N^2 \psi_t \rangle < C$. We will consider the intermediate situation where $E(t) = E_0 + E_1 \cos(2\pi/T)\, t$ and $E_1 > E_0 > 0$, to obtain the following result.

Theorem 5.1. Suppose $a_n \geqslant c\, |n|^{-\nu}$ for some $c > 0$ and $\nu > 3/2$. Then there exists a set of Liouville ω so that the following holds. Given ω

in this set and given E_0, there exists a countable and dense set of values of E_1 in $]E_0, \infty[$ so that for all $\varepsilon > 0$ and $\forall \ell \in \mathbb{Z}$, there exists $C > 0$ so that,

$$\overline{\langle \delta_\ell, U_T^{-m} N^2 U_T^m \delta_\ell \rangle} \geqslant C m^{(2/(\nu + 1/2)) - \varepsilon}$$

Here C depends on ε, ℓ, E_0, ω and E_1.

Note that, as $\nu \to 3/2$, the exponent approaches 1 from below, so that under our hypotheses the motion is always subdiffusive. The lower bound on ν ensures that the evolution leaves the domain of N invariant.

Proof. Comparing (5.4) to (4.3), it is clear that, in order to apply the results of Section 4.3, we need to get a lower bound on the Fourier coefficients $w_n = \bar{w}_{-n}$ of $W_T(x)$:

$$w_n = a_n \int_0^T \exp in \left[E_0 t + \frac{E_1}{\omega} \sin \omega t \right] dt$$

To control the integral, we use a stationary phase argument. Since we assume $E_1 > E_0$, the phase has two stationary points $0 < t_- < t_+ < T$:

$$E_0 + E_1 \cos \omega t_\pm = 0$$

Taking small open intervals $\Delta_\pm \subset]0, T[$ around $t_\pm$ so that $\Delta_+ \cap \Delta_- = \emptyset$, one can construct a smooth partition of unity on $\mathbb{R}/T\mathbb{Z}$ so that

$$\varphi_- + \varphi_0 + \varphi_+ = 1$$

where $\operatorname{supp} \varphi_\pm \subset\subset \Delta_\pm$, $0 \leqslant \varphi_\varepsilon \leqslant 1 \in C^\infty$ and where $\varphi_\pm(t) = 1$ on a small subinterval of $\Delta_\pm$ containing $t_\pm$. Then

$$\int_0^T \exp i2\pi n G(t)\, dt = I_0^n + I_+^n + I_-^n$$

with obvious notations. Since we know that, on the support of φ_0, $|G'(t)|$ is bounded away from 0, an integration by parts shows that $|I_0^n| = O(|n|^{-1})$. For $I_\pm^n$, a stationary phase argument shows that

$$I_\pm^n = \sqrt{\frac{1}{n\, |G''(t_\pm)|}} \exp i2\pi n G(t_\pm) \exp i \frac{\pi}{4} \operatorname{sgn} G''(t_\pm) + o(|n|^{-1/2})$$

Since $G''(t_+) = -G''(t_-) > 0$ one finds

$$\int_0^T \exp i2\pi n G(t)\, dt = \sqrt{\frac{2\pi}{n\omega(E_1^2 - E_0^2)^{1/2}}} \exp i \frac{\pi}{4} \exp i2\pi n G(t_+) \times [1 - i \exp in\Phi(E_1)] + o(|n|^{-1/2})$$

where, for E_0 and ω fixed,

$$\Phi(E_1) = E_0(t_- - t_+) + \frac{2E_1}{\omega} \sin \omega t_-$$

Since Φ is a monotonically increasing continuous function of E_1, mapping $]E_0, \infty[$ onto $]0, \infty[$, one can choose E_1 so that $\Phi(E_1) = 2\pi(p/q)$, with $p, q \in \mathbb{N}$ and q different from 0 modulo 4, which guarantees that

$$\inf_n |[1 - i \exp in\Phi(E_1)]| > 0$$

from which one concludes that

$$\left| \int_0^T \exp i2\pi G(t)\, dt \right| \geqslant C \frac{1}{\sqrt{n}}$$

This implies $w_n \geqslant |n|^{-(\nu + 1/2)}$, so that (5.8) and (4.14) yield the result of the theorem. ▮

REFERENCES

[AK] J. Asch and A. Knauf, Ballistic motion in periodic potentials, mp_arc 97-545.

[B] M. Berry, Incommensurability in an exactly soluble quantal and classical model for a kicked rotator, *Physica D* **10**:369–378 (1984).

[Be] J. Bellissard, Stability and instability in quantum mechanics, in *Trends and Developments in the 80's*, S. Albeverio and Ph. Blanchard, eds. (Bielefeld, 1985).

[BCM] J. M. Barbaroux, J. M. Combes, and R. Montcho, Remarks on the relation between quantum dynamics and fractal spectra, to appear in *Journ. Math. Anal. Appl.* **213**: 698–722 (1997).

[C] J. M. Combes, Connection between quantum dynamics and spectral properties of time evolution operators in *Differential Equations and Applications in Mathematical Physics*, W. F. Ames, E. M. Harrell, and J. V. Herod, eds. (Academic Press, 1993), pp. 59–69.

[CFKS] H. L. Cycon, R. Froese, W. Kirsch, and B. Simon, *Schrödinger Operators*, Springer Verlag (1987).

[CG] G. Casati and I. Guarneri, Non-recurrent behaviour in quantum dynamics, *Commun. Math. Phys.* **95**:121–127 (1984).

[DBF] S. De Bièvre and G. Forni, On the growth of averaged Weyl sums for rigid rotations, *Studia Mathematica*, to appear.

[DK] D. H. Dunlap and V. M. Kenkre, Dynamical localization for a charged particle moving under the influence of an electric field, *Phys. Rev. B* **346**:3625–3633 (1986).

[DJLS] R. del Rio, S. Jitomirskaya, Y. Last, and B. Simon, What is localization?, *Phys. Rev. Lett.* **75**(1):117–119 (1995).

[G] G. Gallavotti, *Elements of Classical Mechanics*, Springer Texts and Monographs in Physics (1983).

[FGP] S. Fishman, D. Grempel, and R. Prange, Localization in an incommensurate potential: An exactly solvable model, *Phys. Rev. Lett.* **49**:833–836 (1982).

[G] I. Guarneri, On an estimate concerning quantum diffusion in the presence of a fractal spectrum, *Europhys. Lett.* **21**(7):729–733 (1993).

[Ho] J. S. Howland, Quantum stability, in *Schrödinger Operators, The Quantum Mechanical many Body Problem*, E. Balslev, ed., Springer Lecture Notes in Physics, Vol. 403 (1992), pp. 100–122.

[IS] F. M. Israilev and D. L. Shepelyanski, Quantum resonance for a rotator in a nonlinear periodic field, *Theor. Math. Phys.* **43**:553–560 (1980).

[J] A. Joye, Upper bounds for the energy expectation in time dependent quantum mechanics, *J. Stat. Phys.* **85**:575–606 (1996).

[K] A. Khinchin, *Continued Fractions*, The University of Chicago Press, Chicago and London (1964).

[L] Y. Last, Quantum dynamics and decompositions of singular continuous spectra, *J. Funct. Anal.* **142**:406–445 (1996).

[N] G. Nenciu, Adiabatic theory: Stability of systems with increasing gaps, *Ann. Inst. H. Poincaré* **67**(4):411–424 (1997).

[O] C. R. De Oliveira, Spectral properties of a simple Hamiltonian model, *Journ. Math. Phys.* **34**(9):3878–3886 (1993).

[P] K. Petersen, On a series of cosecants related to a problem in ergodic theory, *Compositio Mathematica* **26**(3):313–317 (1973).

[RS I] M. Reed and B. Simon, *A Course in Mathematical Physics*, Vol. I, Academic Press (1978).

[RS III] M. Reed and B. Simon, *A Course in Mathematical Physics*, Vol. III, Academic Press (1987).

[S] L. Schwartz, *Analyse*, Hermann, Paris (1970).

[Si] B. Simon, Operators with singular continuous spectrum I: General operators, *Ann. of Math.* **141**:131–145 (1995).